microstructure parameters $\alpha_0^{\cdot}$ and $A_*^{\cdot}$. The coefficients $L_{ij}, F_{ijkl}, M_{ij}, E_{ijkl}$ and AGb^2T^{-1} in (5.15) for the DF are expressed in terms of the macroscopic characteristics of the ensemble of dislocation structures and have a specific value and an explicit physical meaning, and can be determined from the solution of the equations of the model /8, 9/.

REFERENCES

1. SEDOV L.I., Mechanics of a Continuous Medium, 1, Nauka, Moscow, 1973.
2. SEDOV L.I., Mathematical methods of constructing new models of continuous media, Usp. Mat. Nauk, 20, 5, 1965.
3. IVLEV D.D., On the dissipative function in the theory of hardening plastic media, PMM, 31, 2, 1967.
4. IVLEV D.D., On the dissipative function in the theory of plastic media, Dokl. Akad. Nauk, SSSR, 176, 5, 1967.
5. KOLAROV D., BALTOV A. and BONCHEVA N., Mechanics of Plastic Media. Mir, Moscow, 1979.
6. PERZHINA P. and SAWCHUK A., Problems of thermplaticity. Problems of the Theory of Plasticity and Creep, Mir, Moscow, 1979.
7. MROZ Z. and RANIECKI B., A derivation of the uniqueness conditions in coupled thermoplasticity, Intern. J. Eng. Sci., 14, 4, 1976.
8. LUKERCHENKO V.N., On the derivation of the fundamental kinematic relationship of dislocation theory, Dokl. Akad. Nauk SSSR, 274, 1, 1984.
9. LUKERCHENKO V.N., On the theory of deformation hardening of crystalline materials, Fizika i Khimiya Obrab. Materialov, 4, 1983.

Translated by M.D.F.

PMM U.S.S.R., Vol.53, No.5, pp.649-654, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

ON LIMIT SURFACE LOADS IN THE THEORY OF PLASTICITY*

O.O. BARABANOV

Within the framework of quasistatic plasticity theory, the specific features of surface tangential loading is demonstrated by simple examples: the possibility of a singular surface discontinuity, and the absence of convergence of limit load coefficients for an arbitrary unlimited diminution of the period of the plastic composite. The second singularity forces an acknowledgement that the hypothesis /1/ and its subsequent verification are false in the case of tangential surface loads.

1. ***Antiplane motions. Singular surface breakdown.*** We confine ourselves to the examination of rigidly plastic bimaterials in the antiplane and plane cases. The inhomogeneity will be given by using the periodic function $\tau(y)$, defined in the periodicity cell $Y = (0, 1)^2$ as follows

$$\tau(y) = \begin{cases} \tau_1, & y \in Y_k \\ \tau_2, & y \in Y \setminus Y_k \end{cases}$$

$$Y_k = \{y\colon |2y_i - 1| < k,\ i = 1, 2\},\ 0 < \tau_1 \leqslant \tau_2,\ 0 < k < 1$$

**Prikl.Matem.Mekhan.*, 53, 5, 824-829, 1989

where k, τ_1, τ_2 are fixed numbers. We call the sections $\tau = \tau_1$ of the plane inclusions. Let Q be a plane domain under loading. When Q is a coordinate rectangle, its sides will be denoted thus: L is left, R is right; U is up, and D is down.

Let $Q = (-\alpha, \alpha) \times (-\beta, \beta)$. We put

$$f(x, \nabla u) = \tau(x) |\nabla u(x)|, \quad F(u) = \int_Q f(x, \nabla u)\, dx \tag{1.1}$$

$$Bu = \int_L u\, ds - \int_R u\, ds$$

Here u is an arbitrary antiplane velocity field, F is the dissipation functional, B is the surface load, and ds is the Lebesgue measure along the boundary ∂Q of the domain Q.

The problem of the limit load is to find the limit coefficient $\theta = \theta(f, B)$ of the load B by one of the following formulas (see /3/, say)

$$\theta = \inf \{F(u)/Bu \cdot u \in H^1, Bu > 0\} \tag{1.2}$$

$$\theta = \max \lambda \tag{1.3}$$

where H^1 is a Sobolev space with norm $(\int_Q |u|^2 + |\nabla u|^2 dx)^{1/2}$, the maximum is taken among all $\lambda > 0$ such that the load λB is equilibrated by the allowable stress field $\sigma = (\sigma_1, \sigma_2) \in L^\infty$, i.e.,

$$\int_Q \sigma \nabla u\, dx = \lambda Bu, \quad \forall u \in H^1$$

$$|\sigma(x)| \leqslant \tau(x) \text{ almost everywhere on } Q.$$

We assume α to be such that there is a point x_0 belonging simultaneously to the interior of L and the interior of the inclusion. Let $\Delta_\delta \subset Q$ be an isosceles triangle with middle of the base at the point x_0 whose altitude has the dimension δ^2 and base the dimension 2δ. We define a continuous piecewise-affine function u_δ on Q as follows: $u_\delta(x_0) = \delta^{-1}$, $u_\delta \equiv 0$ on $Q \setminus \Delta_\delta$ and on each of the halves of the triangle Δ_δ separated by the altitude the function u_δ is continued affinely. It can be verified that $Bu_\delta = 1$, $F(u_\delta) \to \tau_1$ for $\delta \downarrow 0$. Then according to (1.2)

$$\theta \leqslant \lim F(u_\delta)/Bu_\delta = \tau_1$$

On the other hand, the load $\tau_1 B$ is equilibrated by the allowable stress field $(-\tau_1, 0)$ Then according to the dual formula (1.3) $\tau_1 \leqslant \theta$ and therefore $\theta = \tau_1$

The sequence u_δ is therefore minimizing for problem (1.2). Its generalized limits should be interpreted as the measure on ∂Q that is singular relative to the measure ds. Simultaneously, a generalized limit of the sequence ∇u_δ exits that is to be understood as the vector-valued measure on the closure of Q. A detailed examination of the questions arising here is beyond the scope of this paper (see also /4, 5/, where the generalized velocity fields are submerged in $L^1(Q) \times L^1(\partial Q)$ and $L^1(Q) \times (L^\infty(\partial Q))'$ which is quite close to the expressed consideration).

2. *The averaging problem.* We examine a sequence of bimaterials with Lagrangian $f_\varepsilon(x, \nabla u) = f(\varepsilon^{-1}x, \nabla u)$. The averaging problem is to confirm the following assertion: a homogeneous material with Lagrangian $f_0(\nabla u)$ exists such that the convergence $\theta(f_\varepsilon, A) \to \theta(f_0, A)$ holds as $\varepsilon \downarrow 0$ in a fairly broad set of combinations of the clamping conditions and loads A.

The operation $f \mapsto f^{\text{hom}}$ of formal averaging is well-known /6/

$$f^{\text{hom}}(\xi) = \inf \int_Y f(y, \xi + \nabla u(y))\, dy, \quad \xi \in \mathbb{R}^2$$

where inf is taken over all Y-periodic functions u for which the interval on the right has meaning. The convergence

$$\theta(f_\varepsilon, A) \to \theta(f^{\text{hom}}, A), \quad \varepsilon \downarrow 0 \tag{2.1}$$

holds /3/ for an arbitrary surface-clamped bounded Lipschitz domain subjected to the bulk loading

$$Au = \int_Q au\, dx \quad (a \in L^\infty(Q))$$

Confirmation of the convergence (2.1) in the case of surface laods is called /3/ "one of the interesting problems of averaging theory". It turns out that (2.1) is violated for surface loads.

*3. **Non-averageability under surface loads.*** We say that η belongs to the set $M_\tau \subset \mathbb{R}^2$ if a Y-periodic stress field $\sigma \in L^\infty(\mathbb{R}^2)$ exists such that almost everywhere

$$|\sigma| \leqslant \tau, \quad \int_Y \sigma\, dy = \eta, \quad \int_{\mathbb{R}^2} \sigma \nabla\varphi\, dy, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2)$$

Remark. Any piecewise-smooth solenoidal field σ with piecewise-smooth surfaces of discontinuity along whose edges a weakened continuity condition is satisfied for σ: $\sigma^+\nu^+ + \sigma^-\nu^- = 0$ (ν is the notation of the external normal) satisfies the last condition.

There is the dual formula /3/

$$f^{\text{hom}}(\xi) = \sup_{\eta \in M_\tau} \xi\eta$$

Let $t = k\tau_1 + (1 - k\tau_2)$. We compare the stress field $\sigma^\eta(y_1, y_2) = (\eta_1 h(y_2), \eta_2 h(y_1))$ to each $\eta \in \mathbb{R}^2$, $|\eta| \leqslant t$ where h is a function of period unity having the form

$$h(r) = \begin{cases} \tau_1/t, & |2r - 1| \leqslant k \\ \tau_2/t, & k < |2r - 1| \leqslant 1 \end{cases}$$

on (0, 1).

It can be confirmed (see the Remark) that for σ^η the conditions listed above are satisfied. Then $\{\eta\colon |\eta| \leqslant t\} \subset M_\tau$ and consequently $t\,|\xi| \leqslant f^{\text{hom}}(\xi)$.

Let $Q = (-1, 1) \times (-\beta, \beta)$, the load B is given by (1.1). For $\varepsilon(n) = 2(2n + 1)^{-1}$ inclusions emerge on L, R and according to Sect.1 a singular surface discontinuity is realized. In particular, $\theta(f_\varepsilon, B) = \tau_1$, $\forall \varepsilon(n)$. On the other hand, according to the estimate obtained for f^{hom}

$$k\tau_1 + (1 - k)\tau_2 \leqslant \theta(f^{\text{hom}}, B)$$

Hence, for $\tau_1 < \tau_2$

$$\underline{\lim}\ \theta(f_\varepsilon, B) < \theta(f^{\text{hom}}, B),$$

which indicates the lack of formal averaging in the case under consideration. The fact that the limit of $\theta(f_\varepsilon, B)$ does not exist for ε tending arbitrarily to zero hence still does not certainly follow. Let us examine the situation in greater detail.

*4. **Relative deviation of inclusions from the boundary as an averaging parameter.*** Let $Q = (-1, 1) \times (-\beta, \beta)$, where β is an irrational number and $0 < \rho = \text{const}$. In this case a sequence $\varepsilon = \varepsilon(n) \downarrow 0$ exists such that the deviation of the inclusion in Q from L, R equals $\rho k\varepsilon/2$ and is not less than $(1 - k)\varepsilon/2$ from U, D. As before, let the load B be given by (1.1). For brevity, we introduce the notation $\theta_\varepsilon = \theta(f_\varepsilon, B)$. We obtain the estimates

$$\theta_\varepsilon^-(\rho) \leqslant \theta_\varepsilon \leqslant \theta_\varepsilon^+(\rho) \tag{4.1}$$

from which it will follow that the limit of θ_ε (in the announced sequence $\varepsilon(n)$) depends on ρ.

For symmetry reasons we will examine just the side L with its nearest neighbourhood in Q. We determine ε from the announced sequence.

Let T be a rectangular strip between L and the inclusion (Fig.1), and T_δ an analogous strip with dimensions δ-greater (see Fig.1). We define a continuous piecewise-affine function u_δ in Q as follows: $u_\delta \equiv 1$ on T, $u_\delta \equiv 0$ on $Q \setminus T_\delta$ and the function u_δ is continued piecewise-affinely on $T_\delta \setminus T$. It can be confirmed that the convergences

$$\int_Q f_\varepsilon(x, \nabla u_\delta)\, dx \to (\tau_1 + \rho\tau_2) k\varepsilon, \quad Bu_\delta \to k\varepsilon$$

hold for $\delta \downarrow 0$.

Therefore $\theta_\varepsilon \leqslant \tau_1 + \rho\tau_2 \overset{\text{def}}{=} \theta_\varepsilon^+(\rho)$

To obtain the left estimate of (4.1) we start from the auxiliary stress field in the trapezoid $ABCD$ in Fig.2 that is symmetrical about the y_1 axis. In effect we set $\sigma_1 = -cy_1^{-1}$, $\sigma_2 = -cy_2 y_1^{-2}$. For such a stress field $\operatorname{div}\sigma = 0, \sigma\nu = 0$ on BC and AD. If $\lambda > 0$ is given, then for an appropriate c we will have $\sigma\nu = \lambda$ on AB and $\sigma\nu = -(|AB| / |CD|)\lambda$ on CD.

Here within the limits of the trapezoid

$$\max |\sigma| = \lambda\,(1 + (|CD| - |AB|)^2/(2h)^2)^{1/2} \tag{4.2}$$

where h is the height of the trapezoid.

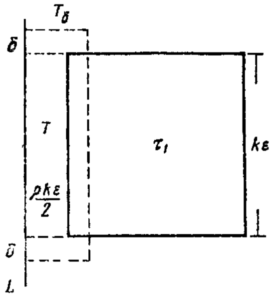

Fig.1

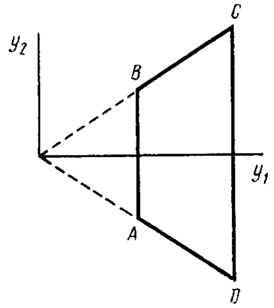

Fig.2

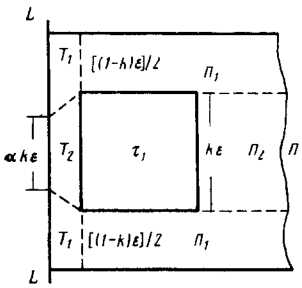

Fig.3

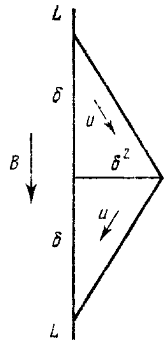

Fig.4

We now construct a piecewise-continuous solenoidal field $\sigma_{\alpha\lambda}$ (in the sense of distributions) in the horizontal rectangular strip $\Pi \subset Q$ of width ε (Fig.3). Namely, we construct a stress field in the trapezoids T_1, T_2 (Fig.3) by the above-mentioned method such that the equality $\sigma_{\alpha\lambda}\nu = \lambda$ is satisfied on L (we consider the two halves of T_1 as one trapezoid; α is the ratio of the bases of T_2, $\alpha < 1$). Then on the bases on the trapezoids T_1 and T_2 interior relative to Q we obtain respectively

$$\sigma_{\alpha\lambda}\nu = -(1 - \alpha k)/(1 - k), \quad \sigma_{\alpha\lambda}\nu = -\alpha\lambda$$

We, respectively, set

$$\sigma_{\alpha\lambda} = (-(1 - \alpha k)\,\lambda/(1 - k),\ 0), \quad \sigma_{\alpha\lambda} = (-\alpha\lambda,\ 0) \tag{4.3}$$

in the strips Π_1, Π_2 resting on T_1, T_2.

Retaining the notation, we continue the field $\sigma_{\alpha\lambda}$ periodically from Π into the subdomain of Q of the form $\Pi + \varepsilon m e_2$, where e_2 is the basis vector of the direction x_2 and m is an integer. We set $\sigma_{\alpha\lambda} = (-\lambda,\ 0)$ on the remaining horizontal strips abutting on U, D (not intersecting the inclusions in conformity with the selection of the sequence $\varepsilon(n)$). It is seen that the field $\sigma_{\alpha\lambda}$ constructed equilibrates the load λB in Q. Then it follows from (1.3), (4.2), and (4.3) that

$$\theta_\varepsilon^-(\rho) \overset{\text{def}}{=} \max_{\alpha>0} \min\left\{\frac{\tau_1}{\alpha},\ \frac{1-k}{1-\alpha k}\left(1 + \left(\frac{1-\alpha}{\rho}\right)^2\right)^{-1/2}\tau_2\right\}$$

can be taken as $\theta_\varepsilon^-(\rho)$.

The functions $\theta_\varepsilon^-(\rho)$, $\theta_\varepsilon^+(\rho)$ are continuous, strictly increasing functions of ρ ($\rho > 0$) For $\rho \downarrow 0$ they have a common limit τ_1. Hence follows the derivation of the dependence of the limit θ_ε on the parameter ρ that characterizes the relative location of the domain and inclusion boundaries.

5. *Plane motions*. The following example is analogous to the preceding one in meaning. Let $Q = (-1,\ 1)^2$ We set

$$f(x, e(\mathbf{u})) = \sqrt{2}\,\tau(x)\,|e(\mathbf{u})(x)|, \quad F(u) = \int_Q f(x, e(\mathbf{u}))\,dx \tag{5.1}$$

$$B\mathbf{u} = \int_U u_1\,ds - \int_L u_2\,ds - \int_D u_1\,ds + \int_R u_2\,ds$$

$$(e(\mathbf{u})(x))_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u\,/\partial x_i), \quad |e(\mathbf{u})|^2 = e_{11}^2 + 2e_{12}^2 + e_{22}^2$$

Here $\mathbf{u} = (u_1, u_2)$ is the velocity field, and $e(\mathbf{u})$ is the strain rate tensor. The surface load (5.1) is tangential and, obviously, selfequilibrated.

The limit coefficient $\theta = \theta(f, B)$ is found from one of the following formulas* in /7/ (these same formulas, with respect to the general case, are the original in /2/ also):

$$\theta = \inf\{F(\mathbf{u}) : \mathbf{u} \in \mathbf{H}^1,\ \operatorname{div}\mathbf{u} = 0, B\mathbf{u} = 1\}, \quad \theta = \max\lambda$$

where the maximum is taken among all values $\lambda > 0$ such that the load λB is equilibrated by the allowable stress field $\sigma = (\sigma_{ij}) = (\sigma_{ji}) \in \mathbf{L}^2(Q)$, i.e.,

$$\int_Q \sum_{1 \leqslant i, j \leqslant 2} \sigma_{ij} e(u)_{ij}\,dx = \lambda B\mathbf{u}, \quad \forall \mathbf{u} \in \mathbf{H}^1$$

$(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 \leqslant 4\tau^2$ almost everywhere on Q.

As before, we set $f_\varepsilon(x, \xi) = f(\varepsilon^{-1}x, \xi)$, $\theta_\varepsilon = \theta(f_\varepsilon, B)$.

Non-averageability can also be shown in this case. Namely, the strict inequality

$$\lim \theta_{\varepsilon'} < \underline{\lim}\, \theta_{\varepsilon''} \tag{5.2}$$

can be obtained for two different sequences $\varepsilon' \downarrow 0$ and $\varepsilon'' \downarrow 0$.

We first consider the auxiliary problem of a limit load in the domain $Q = Y \setminus Y_k$. Let the interior surface of the domain Q be free and the exterior subjected to the tangential loading (5.1). We denote the appropriate limit coefficient by τ_* $(0 < \tau_*)$ and σ_0 is the allowable stress field equilibrating the load $\tau_* B$. We continue the field σ_0 to zero on Y_k and then Y-periodically to the whole plane. We denote the field obtained by σ_*.

Let $Q = (-1, 1)^2$, $0 < \tau_1 < \tau_*$, and the load B is given by (5.1). We examine the sequence $f_{\varepsilon'}$, where $\varepsilon' = 2(2n+1)^{-1}$. In this case the inclusions emerge on the boundary of Q and a singular surface discontinuity is realized.

Indeed, we take a discontinuous velocity field localized on an inclusion emerging from the boundary and consisting of three rigid parts slipping relative to each other (Fig.4). According to a known method reinforced by the proof /4, 5/

$$\theta_{\varepsilon'} \leqslant \tau_1 \sum_i |[\mathbf{u}]|\, l_i / B\mathbf{u}$$

where $[\mathbf{u}]$ is the velocity jump on the interfacial boundary of the rigid parts and l_i is the length of the appropriate (i-th) piece of the interfacial boundary. Elementary calculations result in the estimate $\theta_{\varepsilon'} \leqslant \tau_1(1 + \delta^2)$. Passing to the limit as $\delta \downarrow 0$ (for a fixed ε'), we obtain that $\theta_{\varepsilon'} \leqslant \tau_1$. The allowable stress field $\sigma_{11} = 0$, $\sigma_{12} = \tau_1$, $\sigma_{22} = 0$ equilibrating the load $\tau_1 B$ results in the estimate $\tau_1 \leqslant \theta_{\varepsilon'}$. Therefore, $\theta_{\varepsilon'} = \tau_1$, $\forall \varepsilon'$.

Now we examine the sequence $f_{\varepsilon''}$, $\varepsilon'' = n^{-1}$ It is seen that the stress field $\sigma_{\varepsilon''}(x) = \sigma_*(x/\varepsilon'')$ is equilibrating to the load $\tau_* B$ and allowable. Therefore, $\tau_* \leqslant \theta_{\varepsilon''}$, $\forall \varepsilon''$.

As a result, by choosing $\tau_1 < \tau_*$ we obtain (5.2).

Formally speaking, the example constructed is not a counterexample to theorems of /2/ that affirmatively solve (without proof) the averaging problem in the whole load spectrum, but for domains with smooth boundaries. However, it is obvious that the situation is not in the presence of angles for the domains. Thus, a smooth expansion of the domain (above D and under U) in the antiplane examples of Sects.3, 4 conserves the fact of non-averageability. The effect that is called the singular surface breakaway above explains the non-averageability of plastic media extremely simply. Let the domain geometry be sufficiently "good". When softer inclusions emerge on the tangentially loaded boundary the limit load is a function of the inclusions flow exclusively. In other cases of the relative location of the inclusions and boundary, the limit load depends essentially on the flow characteristics of the harder matrix of the composite also. The examples presented confirm and refine this reasoning.

(*Barabanov, O.O., On equivalent formulations of the limit elastic-plastic problem. Deposited in VINITI January 31, 1986. No.729-V86 Dep., Vladimir, 1986.

REFERENCES

1. SUQUET P., Analyse limite et homogénéisation, C.R. Acad. Sci. Paris, Ser. II, Mec., 296, 18, 1983.
2. DEMENGEL F. and TANG Qi, Homogénéisation en plasticité, C.R., Acad. Sci . Paris, Ser. I, 303, 8, 1986.
3. ZHIKOV V.V., Averaging of functionals of calculus of variations and elasticity theory, Izv. Akad. Nauk. Ser. Mat., 50, 4, 1986.
4. SEREGIN G.A., Expansion of the variational formulation of the problem for a rigidly plastic medium to a velocity field with slip-type discontinuities, PMM, 47, 6, 1983.
5. KAMENYARZH YA.A., Conditions on surfaces of discontinuity in a rigidly plastic analysis, Dokl. Akad. Nauk SSSR, 286, 3, 1986.
6. BERDICHEVSKII V.L., Variational Principles of the Mechanics of a Continuous Medium. Nauka, Moscow, 1983.
7. KAMENYARZH YA.A., On formulations of problems of the theory of ideal plasticity, PMM, 47, 3, 1983.

Translated by M.D.F.

PMM U.S.S.R.,Vol.53,No.5,pp.654-659,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

A CRACK ON THE INTERFACIAL BOUNDARY OF PRESTRESSED ELASTIC MEDIA*

V.B. ZELENTSOV and L.M. FILIPPOVA

The plane problem of the equilibrium of a piecewise-homogeneous body weakened by a crack located on the interfacial boundary of the materials and under uniform loading is considered. There are initial stresses in the body that act in the direction of the interfacial boundary. The solution of the problem is found by reduction to a system of singular integral equations. It is established that exactly as in an analogous problem without taking account of the initial stresses /1-3/, the solution near the crack tip is rapidly oscillating in nature, where the oscillation zone is broadened as the initial compression increases.

1. We consider a piecewise-homogeneous elastic body consisting of two half-planes interconnected along the whole interfacial boundary $y = 0$ with the exception of the segment $|x| < 1$ which is a rectilinear crack in the form of an infinitely thin slit. Here x, y are dimensionless coordinates referred to the crack length a. The body is subjected to a preliminary homogeneous finite strain for which there are no stresses on lines parallel to the x axis. The crack edges are loaded by uniform pressure p and a uniform shearing load of intensity τ. The strain caused by the loading of the crack edges is assumed to be small, and consequently, we use linearized equilibrium equations for a prestressed medium to solve the problem /4/.

For non-linearly elastic materials of general form the solution of the problem gives rise to serious technical difficulties. Consequently, we will investigate specific models of materials. It is assumed in this section that the materials filling the lower and upper half-planes are incompressible and described by the Mooney model /4, 5/ with shear modulus G_1 in the lower $y < 0$ half-plane and shear modulus G_2 in the upper $y < 0$ half-plane.

The mathematical formulation of the problem constains boundary conditions on the line $y = 0$

$$\begin{gathered} u_1 = u_2, \quad v_1 = v_2, \quad \theta_{yy1} = \theta_{yy2}, \quad \theta_{yx2} = \theta_{yx1}, \quad 1 < |x| < \infty \\ \theta_{yy1} = \theta_{yy2} = -p, \quad \theta_{yx1} = \theta_{yx2} = \tau, \quad |x| \leqslant 1 \end{gathered} \tag{1.1}$$

Prikl.Matem.Mekhan.,53,5,830-836,1989